Prisoners' Other Dilemma[*]

Matthias Blonski[†] Giancarlo Spagnolo[‡]

December 2004
First draft: March 2001

Abstract

We develop an intuitive measure of the "risk of being cheated upon" in the discounted infinitely repeated Prisoner's Dilemma, based on – but distinct from Harsanyi and Selten's (1988) concept of *risk dominance.* We then show that co-operation as equilibrium is often not plausible: For a significant subset of the payoff-discount factor parameter space, *all* co-operation equilibria are *risky.* We derive an easy-to-calculate critical level for the discount factor δ^* below which this happens, and argue it is a better measure for the "likelihood" of cooperation than the critical level $\underline{\delta} < \delta^*$ at which cooperation is supportable in equilibrium. Sufficient conditions are provided for cooperation equilibria to be *risk perfect,* i.e. not to be risky at any subgame. Our results apply to other games sharing the strategic structure of the repeated Prisoner's Dilemma (repeated oligopolies, relational-contracting models, etc.). We illustrate our main result for collusion equilibria in the repeated Cournot duopoly. *JEL Classification Numbers*: C72, L13, L14, M50.

---

[*] We appreciate suggestions and remarks by Geir Asheim, Eric van Damme, Christian Ewerhart, David Levine, Klaus Ritzberger and Reinhard Selten and various seminar participants in Berkeley, Bonn, Frankfurt, Mannheim, Stanford, Stockholm, Tilburg, and Zürich. This project has been supported by grants from the DFG (BL418/4-1) for Blonski, and from the European Commission (EGEMTPS/RTN) for Spagnolo.

[†] Department of Economics, J.W.Goethe University, Frankfurt am Main, 60054 Frankfurt a.M., Germany; telephone: +49-(0)69-798-23858; e-Mail: *blonski@wiwi.uni-frankfurt.de*

[‡] Department of Economics, University of Mannheim, and C.E.P.R. e-Mail: *gianca@uni-mannheim.de*

1 Introduction

Tacit collusion, relational contracts, monetary policies, financial arrangements, international agreements and social relations are among the economic and social institutions that have been modelled as equilibria of dynamic games with the strategic structure of a repeated Prisoner's Dilemma. A common feature of all these models is that the conditions for an agreement being supportable in equilibrium are independent of the payoff one party obtains when another party "cheats". After the initial description, payoffs obtained when being "cheated upon" by other agents drop out of the model, since only incentive compatibility conditions of potential deviators ("cheaters") are considered, and these only consider their own incentives. We are convinced that real world parties do care about what would happen if their partner defected from the jointly agreed upon strategy profile, and that these considerations should not be assumed away from our models. By turning attention to strategic risk we propose a simple criterion for all models of the repeated Prisoner's Dilemma kind that reflects a participant's concern that co-operation may fail.

To illustrate the idea consider the following version of the Prisoner's Dilemma (PD), denoted by Γ, where $a < \frac{7}{3}$.

Γ	c	d
c	3 3	$\frac{10}{3}$ a
d	a $\frac{10}{3}$	$\frac{7}{3}$ $\frac{7}{3}$

Denote by $\Gamma(\delta)$ the infinitely repeated PD with common discount factor δ where co-operation can be supported as subgame perfect equilibrium as long as players' discount factor is above the lower bound $\underline{\delta}$ that equalizes short run gains and (maximal) long run losses from defecting (e.g. Friedman, 1971; Fudenberg and Maskin, 1986; Abreu, 1988). In this example, co-operation is supportable as equilibrium behavior in $\Gamma(\delta)$ as long as $\delta \geq \underline{\delta} := \frac{1}{3}$.[1] In many applications this lower bound $\underline{\delta}$ has been used as an (inverse) index of how plausible or likely co-operation is in a given environment. The literature on renegotiation-proofness has shown that this conclusion does not change if we require continuation strategies not to be Pareto-dominated (van Damme, 1989; Farrell and Maskin, 1989).

In our view the literature on repeated games has too much been preoccupied with Folk-Theorem results and co-operation equilibria, and thereby failed to recognize the

---

[1] Then discounted payoffs from playing cooperation indefinitely, $\frac{3}{1-\delta}$, offset those from defecting unilaterally and being kept at the minimax thereafter, $\frac{10}{3} + \frac{7}{3}\frac{\delta}{1-\delta}$.

potential fragility of co-operation. As mentioned, the lower bound $\underline{\delta}$ does not depend on the parameter a of the stage game, while we have the strong feeling that this parameter would and should influence players' propensity to cooperate. To get an immediate taste, note that if parameter a goes to $-\infty$ it appears to be most risky ever to co-operate even for a single period. In this article we formalize the consequences of variations in parameter a in the infinitely repeated discounted Prisoner's Dilemma, and in games with analogous features. We argue that in many cases (i.e. if $\delta \geq \underline{\delta}$) the original Prisoner's Dilemma cannot be fixed so easily by repeated interaction since players are susceptible to strategic risk. We characterize this problem in terms of payoff parameters and discount rates and whenever it arises we call it Prisoners' Other Dilemma.[2]

To see that stage game parameter a should be taken into account, and to give a hint on how this can be done in a precise way, we look at $\Gamma(\delta)$ for $a = -\frac{14}{3}$ and $\delta = \frac{2}{3} > \frac{1}{3} = \underline{\delta}$. Suppose for the moment that players only consider the following two pure strategies of the repeated stage game:

c^* : Co-operate as long as no player defects, defect forever otherwise.
d^* : Defect forever.

The so defined $2 \times 2-$ game Γ^* is given by

Γ^*	c^*	d^*
c^*	9 9	8 0
d^*	0 8	7 7

,

where numbers are discounted sums of payoffs, and reflect the strategic reasoning of players playing $\Gamma(\delta)$ but restricting attention only to these two strategies c^* and d^*.[3] The game Γ^* is sometimes called "stag hunt" game. It has two strict pure strategy equilibria $C^* := (c^*, c^*)$ and $D^* = (d^*, d^*)$ where D^* is strictly payoff dominated by C^*.[4] Which of those two equilibria do we expect to be selected? While $C^* := (c^*, c^*)$ is the payoff-dominant equilibrium, a cautious player might prefer to play d^* to avoid extreme losses, for example if the opponent makes mistakes. But even rational (and never failing) players who just are not sure about their opponent's beliefs on their own rationality, or have any higher order doubt, may prefer d^* over c^*. Moreover, pre-play

---

[2] At a later stage in this article (section 5.1) we will learn in more detail why Prisoners' Other Dilemma in contrast to Prisoner's Dilemma (note the different use of the ') is formulated as a joint problem. For now we just keep in mind that the original PD is generated by individual incentives while POD will be caused by mutual strategic risk.

[3] Note that picking a different equilibrium corresponds to a different game Γ^*.

[4] There is another (weak) mixed equilibrium which is not of interest here.

communication does not help to coordinate in this game since a player planning to play d^* has an incentive to convince his opponent to play c^*.[5]

Harsanyi and Selten, in their influential book on equilibrium selection (1988, subsequently abbreviated with HS) introduce *risk dominance* as selection criterion and demonstrate that in $2 \times 2-$games with two strict pure strategy equilibrium points the risk dominance relation (in contrast to the payoff dominance relation) is invariant under transformations on payoffs preserving the best reply structure.[6] According to HS risk dominance in these very simple games can be evaluated by comparing the so called Nash-products of the two equilibria.[7]

In this article we propose a new concept to measure strategic risk in the repeated Prisoner's Dilemma that relies on the simple and intuitive notion of risk dominance in the $2 \times 2-$game Γ^*. In order to measure more generally the "riskiness to co-operate" in the discounted, infinitely repeated Prisoner's Dilemma we compare *all* co-operation equilibria with the defection equilibrium D^*. The all-defect equilibrium D^* is a natural reference point as it is the unique "safe" equilibrium, in the sense that it represents the only mode of behavior (for both players) that avoids payoff a under every possible history and thereby guarantees the maxmin payoff against any other strategy.

The strategic risk comparison between all co-operation equilibria with the defection equilibrium D^* opens the door for our main characterization result (Theorem 1). There we characterize for which PD-type games this "strategic risk problem" is relevant for *all* equilibria supporting co-operation, and call this problem *Prisoners' Other Dilemma*. We characterize the set of PD-games parameters featuring Prisoners' Other Dilemma, and show that it is never empty – in particular, for no given discount factor.

More specifically, we show that for players susceptible to this sort of strategic risk, the lower bound $\underline{\delta}$ is no good indicator for the plausibility of co-operative behavior, as frequently used in the applied literature. Within our main result, we propose an alternative easy-to-derive lower bound δ^* for discount factors, with $\delta^* > \underline{\delta}$, to be used as a novel and better tool that also depends on parameter a and reflects the riskiness to co-operate, besides the incentive to defect. In particular, for asymmetric PD-games the incentive to defect may vary from player to player, i.e. $\underline{\delta}_1 \neq \underline{\delta}_2$ while δ^* does not depend

---

[5] This point was raised by Aumann (1990) who used this same example to motivate his objection against the self enforcing nature of Nash equilibria.

[6] Payoff differences (incentives to switch to another strategy) for any given opponent's behavior remain unchanged.

[7] The Nash product measures is the product of both players disincentives not to behave according to the equilibrium under consideration. In our example D^* risk dominates C^* because $(7-0)(7-0) = 49 > 1 = (9-8)(9-8)$. We picked these numbers because Γ^* has made a career into game theory textbooks (see for example Fudenberg and Tirole 1991, p. 21) as an example for a game where the selection criteria payoff dominance and risk dominance conflict with each other.

on the player since δ^* reflects a "mutual strategic risk". In our introductory example, it turns out that $\delta^* = \frac{11}{12} > \frac{1}{3} = \underline{\delta}$.

Moreover, we argue that if players are susceptible to strategic risk, an equilibrium should be considered sufficiently safe only if strategic risk is not a problem in any of its (out-of-equilibrium) subgames. We call this property *risk perfection*, and provide sufficient conditions under which it is satisfied for co-operation equilibria (Theorem 3). We show that these conditions include the relevant punishment strategies being used in the (theoretical and applied) literature.

The strategic risk measure we propose is different but based on HS's concept of risk dominance (which cannot be defined for infinite games like the repeated PD). Although HS favor payoff-dominance over risk dominance as selection criterion in their book, the theoretical and experimental support towards risk dominance has increased since then.[8] Theoretical support has been offered by the evolutionary game theory literature (see for example Kandori, Mailath and Rob, 1993; and Young 1993) and by the literature on global games considering perturbations of payoff parameters (starting with Carlsson and van Damme, 1993a). Experimental evidence also tends to support risk and security if they conflict with the payoff criterion (a well known reference is van Huyck, Battalio, and Beil, 1990). A more systematic experimental investigation including payoff parameter variations was recently performed by Schmidt et al. (2003), whose main finding, summarized in the abstract, is that "changes in risk dominance significantly affect play of the subjects whereas changes in payoff dominance do not."

We focus in this paper on the discounted infinitely repeated Prisoner's Dilemma. However, definitions and results directly apply to many other games that share its strategic features, including repeated oligopoly and public good games, and implicit/relational contracting models. We elaborate further on this in the final section applying our results to the textbook example of a repeated Cournot-collusion game with linear demand.

---

[8] In this context it is noteworthy, that even Harsanyi and Selten did not agree on this point and Harsanyi (1995) later came up with an alternative selection theory where he decided to reverse priority and favour risk dominance over payoff dominance.

2 Strategic risk

In this section we consider the following standard symmetric PD stage game Γ characterized by payoff parameters[9] a, b, c, d where $b > c > d > a$ and $2c > b + a$,[10]

Γ	c	d
c	c c	b a
d	a b	d d

.

Denote the vector of payoff parameters by $\lambda = (a, b, c, d)$ and together with a common discount factor by $s = (\lambda, \delta) = (a, b, c, d, \delta)$ and call $\Gamma(\delta)$ the related infinitely repeated game with common discount factor δ. We call an equilibrium φ that supports indefinite co-operation as its equilibrium outcome path a *co-operation equilibrium.* Denote by ω the max-min equilibrium or *defection equilibrium* where each player plays the unique stage game dominant strategy forever.

For each co-operation equilibrium φ, introduce the game Γ_φ – subsequently called the *φ–formation of* $\Gamma(\delta)$. The φ–formation Γ_φ is a substructure of $\Gamma(\delta)$ capturing the strategic considerations of players restricting their attention to the binary subset of the strategy space defined by the two equilibria φ and ω. More precisely, Γ_φ is the 2×2–game defined by the strategy space $\{\varphi_i, \omega\}$ where $\varphi = (\varphi_1, \varphi_2)$ and $\omega = (\omega, \omega)$ are the equilibrium strategy profiles for both players.[11] The bimatrix-form of Γ_φ is given by

Γ_φ	φ_2	ω
φ_1	$\frac{c}{1-\delta}$ $\frac{c}{1-\delta}$	$b + \delta V_{2\omega}$ $a + \delta V_{1\varphi}$
ω	$a + \delta V_{2\varphi}$ $b + \delta V_{1\omega}$	$\frac{d}{1-\delta}$ $\frac{d}{1-\delta}$

,

where $V_{i\xi}$ for $i\xi \in \{1\varphi, 1\omega, 2\varphi, 2\omega\}$ are the equilibrium payoffs of the corresponding continuation games.[12]

---

[9] In section 5.1 we allow for parameter asymmetries and find that definitions and results are qualitatively unchanged. In order to economize on notation we use c, d as lables for the stage game strategies as well as for payoff parameters as long as there is no confusion.

[10] The first parameter restriction implies the PD while the second restriction excludes cases where indefinite co-operation is not efficient since patient players can improve by defecting alternately.

[11] In the defect equilibrium both players use the same strategy. As long as this does not cause confusion we identify the strategy and the corresponding equilibrium profile with the same symbol ω.

[12] Since continuation payoffs are always $\frac{c}{1-\delta}$ or $\frac{d}{1-\delta}$ if both players pick φ or ω we simplify the notation. For example $V_{1\omega}$ is the continuation payoff of player 1 playing strategy ω if player 2 plays φ_2

The central question of the present inquiry is: Under which circumstances – i.e. for which parameter constellations s – *all* co-operation equilibria are risk dominated within Γ_{φ} by the defection equilibrium ω?[13]

Denote by $\psi = (\psi, \psi)$ the 'grim-trigger'- or 'Nash reversion'-punishment co-operation equilibrium where every player responds to any deviation from co-operation by defecting forever.

For $\delta < \underline{\delta} := \frac{b-c}{b-d}$ the co-operation equilibrium set is empty since

$$b + \delta V_{i\omega} \geq b + \delta V_{i\psi} = b + \frac{\delta d}{1-\delta} > \frac{c}{1-\delta},$$

and even the most severe punishment cannot support indefinite co-operation as equilibrium point of $\Gamma(\delta)$. To rule out these uninteresting cases and cases where co-operation can only be supported by weak equilibria let us assume from now $\delta > \underline{\delta}$ and denote the respective (open set) parameter space by

$$S := \left\{ s = (a, b, c, d, \delta) \left| b > c > d > a,\ 2c > b + a \text{ and } \underline{\delta} < \delta < 1 \right. \right\}.$$

Definition 1 *Let Φ denote the set of co-operation equilibria φ such that Γ_{φ} has two strict pure-strategy equilibrium points, i.e.*

$$\begin{aligned} u_i(\varphi) &: = \frac{c}{1-\delta} - b - \delta V_{i\omega} > 0 \qquad \textit{and} \\ v_i(\varphi) &: = \frac{d}{1-\delta} - a - \delta V_{i\varphi} > 0, \end{aligned}$$

for $i = 1, 2$.

From here we restrict attention to the set Φ of strict co-operation equilibria since weak equilibria are even more "risky" in the sense to be defined now. The set Φ is always non-empty because the corresponding maximal payoff-differences $u(\psi), v(\psi)$ for the grim-trigger-punishment do not depend on i and for our parameter restrictions are always strictly positive

$$\begin{aligned} u(\psi) &= \frac{c}{1-\delta} - b - \frac{d\delta}{1-\delta} = \frac{\delta(b-d) - (b-c)}{1-\delta} > 0 \qquad \text{and} \\ v(\psi) &= \frac{d}{1-\delta} - a - \frac{d\delta}{1-\delta} = d - a > 0. \end{aligned}$$

---

[13] Although our main interest is to explore favorable conditions for co-operation obviously the same question can be asked for every other equilibrium of the repeated PD game, also for inefficient equilibria. We will demonstrate in the extensions that the essence of the results do not change or even aggravate by moving away from full cooperation.

Definition 2 *We call a co-operation equilibrium* $\varphi \in \Phi$ *"risky" if* within the $2 \times 2-$game Γ_φ *it is* strictly risk dominated *by the defection equilibrium* ω, *i.e. iff*

$$u_1(\varphi)\, u_2(\varphi) < v_1(\varphi)\, v_2(\varphi).$$

Correspondingly, for weak inequality we use the notion *weak risk dominance within the* $2 \times 2-$*game* Γ_φ. As Harsanyi and Selten we call $u_1(\varphi)\, u_2(\varphi)$ and $v_1(\varphi)\, v_2(\varphi)$ 'Nash-products' of the corresponding equilibria.

Definition 3 *We call* $\rho(s,\varphi)$ *the "riskiness" of co-operation equilibrium* φ *in* $\Gamma(\delta)$, *where*

$$\begin{aligned} \rho(s,\varphi) \quad : &= v_1(\varphi)\, v_2(\varphi) - u_1(\varphi)\, u_2(\varphi) \\ &= \left(\frac{d}{1-\delta} - a - \delta V_{1\varphi}\right)\left(\frac{d}{1-\delta} - a - \delta V_{2\varphi}\right) \\ &\quad - \left(\frac{c}{1-\delta} - b - \delta V_{1\omega}\right)\left(\frac{c}{1-\delta} - b - \delta V_{2\omega}\right). \end{aligned}$$

For any given co-operation equilibrium φ of a discounted PD-supergame $\Gamma(\delta)$ definition 3 can be easily applied to verify whether the equilibrium is risky. The same can be done for co-operation equilibria of other discounted supergames with analogous strategic features.

Although risk dominance in the restricted game Γ_φ is simple to define it is known that for games with more than two strategies the risk dominant equilibrium in the restricted game Γ_φ may not coincide with Harsanyi and Selten's more general definition of risk dominance as pairwise comparison between two equilibrium points[14]. HS's more general definition of risk dominance is based on the so called *bicentric prior* and the *tracing procedure* and is difficult to check for games with many strategies. This might explain why it rarely found its way into the applied literature with the exception of $2 \times 2-$games with two strict equilibrium points (see HS, Ch. 3). While the bicentric prior and the tracing procedure have ever been controversial among game theorists they offer no clue in our context since HS defined them only for finite games. The game theoretic literature has not yet offered a generalization of risk dominance to infinite games[15]. We provide further motivation and a more detailed discussion of the relationship between the present concept and HS in the appendix.

---

[14] For examples see van Damme and Hurkens (1998, 1999) and Carlsson and van Damme (1993b).

[15] We learned from Klaus Ritzberger that a fundamental mathematical difficulty does not allow a direct generalization of the tracing procedure to infinite games. The idea of the tracing procedure relies on connecting strategy profiles by a generically unique continuous path representing continuous adjustment of beliefs. The problem is that this continuous path is neither unique nor even defined if equilibria are not topologically isolated which happens to be the case in infinite games. In finite games, in contrast, equilibria are generically isolated.

3 Characterization of Prisoners' Other Dilemma

Since $c > d$, any co-operation equilibrium φ payoff-dominates defection ω. Our previous definition shows that strategic risk consideration may point to the opposite direction for a particular co-operation equilibrium. Our task in this section is to characterize the set of all strict co-operation equilibria where payoff dominance and strategic risk considerations point to opposite directions and vice versa. We start by establishing an important benchmark.

Proposition 1 *There exists no co-operation equilibrium $\varphi \in \Phi$ which is less risky than the grim trigger equilibrium ψ. Formally,*

$$\rho(s,\psi) = \underline{\rho}(s) := \inf_{\varphi} \rho(s,\varphi).$$

Proof. No co-operation equilibrium φ can be less risky than $\inf_{\varphi} \rho(s,\varphi)$. First, note that for any $\varphi \in \Phi$ the upper bound for the continuation payoff of a player who plays a co-operation equilibrium strategy against a player who always defects is

$$V_{i\varphi} \leq \frac{d}{1-\delta},$$

and that continuation payoffs of players who always defect are bounded by

$$\frac{d}{1-\delta} \leq V_{i\omega} \leq \frac{b}{1-\delta}.$$

Next, to consider strict co-operation equilibria in Φ imposes additional boundaries on $V_{i\varphi}$ and $V_{i\omega}$ given by $V_{i\varphi} < \frac{1}{\delta}\left(\frac{d}{1-\delta} - a\right)$ and $V_{i\omega} < \frac{1}{\delta}\left(\frac{c}{1-\delta} - b\right)$. Only the second inequality is binding, hence together with the boundaries for $\Gamma(\delta)$ we obtain

$$\begin{aligned} V_{i\varphi} &\leq \frac{d}{1-\delta} \text{ and} \\ \frac{d}{1-\delta} &\leq V_{i\omega} < \frac{1}{\delta}\left(\frac{c}{1-\delta} - b\right), \end{aligned}$$

since $b > c > d > a$ implies $\frac{d}{1-\delta} < \frac{1}{\delta}\left(\frac{d}{1-\delta} - a\right)$ and $\frac{b}{1-\delta} > \frac{1}{\delta}\left(\frac{c}{1-\delta} - b\right)$. Then continuation payoffs must satisfy $V_{i\varphi} \leq \frac{d}{1-\delta}$ and $V_{i\omega} \geq \frac{d}{1-\delta}$. For the grim-trigger strategy equilibrium ψ both conditions are binding and hold with equality. This yields

$$\begin{aligned} \underline{\rho}(s) &= \inf_{\varphi} \rho(s,\varphi) \\ &= \left(\frac{d}{1-\delta} - a - \delta\frac{d}{1-\delta}\right)^2 - \left(\frac{c}{1-\delta} - b - \delta\frac{d}{1-\delta}\right)^2 \\ &= (d-a)^2 - \left(\frac{c}{1-\delta} - b - \delta\frac{d}{1-\delta}\right)^2 \\ &= \rho(s,\psi). \end{aligned}$$

■

It is now time to state the main result of this paper. In order to be more precise about parameters, we introduce the following notation.

Definition 4 *Let S^{ω} denote the set of repeated PD-games where* all *strict co-operation equilibria are risky, i.e.*

$$S^{\omega} := \{s \in S \,|\omega \text{ strictly risk dominates } \varphi \text{ within the } 2\times 2 - \text{game } \Gamma_{\varphi} \quad \forall \varphi \in \Phi(s)\} \subset S.$$

Conversely, let S^{φ} denote the set of repeated PD-games where no *strict co-operation equilibrium is risky, i.e.*

$$S^{\varphi} := \{s \in S \,|\varphi \text{ risk dominates } \omega \text{ within the } 2\times 2 - \text{game } \Gamma_{\varphi} \quad \forall \varphi \in \Phi(s)\} \subset S.$$

The following theorem characterizes these parameter sets.

Theorem 1 *(i) All co-operation equilibria of the repeated PD-game $\Gamma(\delta)$ are risky if and only if $\delta < \delta^*$, with $\delta^* := \frac{b-a-(c-d)}{b-a} > \underline{\delta}$, hence*

$$S^{\omega} = \left\{ s \in S \,\middle|\, \delta < \delta^* = \frac{b-a-(c-d)}{b-a} \right\}.$$

(ii) There exist no parameters $s \in S$ such that no co-operation equilibrium is risky, hence $S^{\varphi} = \emptyset$.

Proof. A little calculation shows that the interval $(\underline{\delta}, \delta^*)$ is never empty

$$\begin{aligned} (d-a)(c-d) &> 0 \Leftrightarrow \\ \frac{b-a-c+d}{b-a} &> \frac{b-c}{b-d} \Leftrightarrow \\ \delta^* &> \underline{\delta}. \end{aligned}$$

This implies that for $\delta \in (\underline{\delta}, \delta^*)$

$$\begin{aligned} \delta &< \frac{b-a-c+d}{b-a} \Leftrightarrow \\ d-a &> \frac{c}{1-\delta} - b - \delta\frac{d}{1-\delta} \Leftrightarrow \\ \left(\frac{d}{1-\delta} - a - \delta\frac{d}{1-\delta}\right)^2 &> \left(\frac{c}{1-\delta} - b - \delta\frac{d}{1-\delta}\right)^2 \Leftrightarrow \\ \rho(s,\psi) &= \underline{\rho}(s) > 0. \end{aligned}$$

Since all implications hold in both directions this implies claim (i) of the theorem. To prove claim (ii), define similarly as in proposition 1

$$\bar{\rho}(s) := \sup_{\varphi} \rho(s,\varphi)$$

as the lowest upper bound on the riskiness among all co-operation equilibria. It remains to show that $\bar{\rho}(s)$ is strictly positive $\forall s \in S$. By the boundaries given in the proof of proposition 1 we know that

$$\inf_{\varphi}\left(\frac{c}{1-\delta}-b-\delta V_{1\omega}\right)\left(\frac{c}{1-\delta}-b-\delta V_{2\omega}\right)=0$$

and that

$$\sup_{\varphi}\left(\frac{d}{1-\delta}-a-\delta V_{1\varphi}\right)\left(\frac{d}{1-\delta}-a-\delta V_{2\varphi}\right)\in\left[(d-a)^2,\left(\frac{d-a}{1-\delta}\right)^2\right].$$

This together yields

$$\begin{aligned}\bar{\rho}(s) &= \sup_{\varphi}\left(\frac{d}{1-\delta}-a-\delta V_{1\varphi}\right)\left(\frac{d}{1-\delta}-a-\delta V_{2\varphi}\right)\\ &\quad -\inf_{\varphi}\left(\frac{c}{1-\delta}-b-\delta V_{1\omega}\right)\left(\frac{c}{1-\delta}-b-\delta V_{2\omega}\right)\\ &\geq (d-a)^2-0>0.\end{aligned}$$

■

As already mentioned in the introduction we label *Prisoners' Other Dilemma* the problem that incentive compatible co-operative behavior ($\delta > \underline{\delta}$) may be considered too risky to “fix Prisoner's (original) Dilemma”. The theorem tells us exactly when players susceptible to strategic risk are unable to overcome the original dilemma by building up a “co-operative relationship”. The theorem also tells us that there exists no discounted repeated PD-game for which this “other dilemma” disappears altogether. There are always some risky co-operation equilibria. Intuitively, one obtains the more risky co-operation equilibria by letting players be more “forgiving”, i.e. try to start co-operative behavior although the opponent defected in the past. In equilibrium, however, this cannot be done too frequently.

The following corollary follows immediately from the theorem. It points to stage game parameter constellations where Prisoners' Other Dilemma tends to be most serious.

Corollary 1 *For a very large payoff-difference $b-a$ or a very small difference $c-d$ all co-operation equilibria are risky for any discount factor $\delta < 1$. Formally,*

$$\lim_{b-a\to\infty}\delta^*=\lim_{c-d\to 0}\delta^*=1.$$

One might think regarding the theorem that after all the strategic risk problem tends to disappear for very patient players. The following proposition has the flavor of an 'anti-Folk theorem for strategic risk' and shows that for any discount factor $\delta < 1$

and appropriately chosen payoff parameters all co-operation equilibria are risky. Even worse, by choosing the payoff parameter a sufficiently low the riskiness of all co-operation equilibria can be made arbitrarily large.

Proposition 2 *For every* $\delta < 1$ *there exist payoff parameters* λ *with* $s = (\lambda, \delta) \in S^{\omega}$ *such that all co-operation equilibria of* $\Gamma(\delta)$ *are risky. Moreover, for any given riskiness* $\rho > 0$ *there exist payoff parameters* λ *with* $s = (\lambda, \delta) \in S^{\omega}$ *such that all co-operation equilibria have at least riskiness* ρ*.*

Proof. Payoff parameter a is not bounded from below. Hence

$$\underline{\rho}(s) = (d-a)^2 - \left(\frac{c}{1-\delta} - b - \delta\frac{d}{1-\delta}\right)^2$$

goes to infinity for $a \to -\infty$. This implies both statements of the proposition. ■

4 Risk Perfection

The idea that in a repeated Prisoner's Dilemma game $\Gamma(\delta)$ players might consider a co-operation equilibrium as 'too risky' – although it Pareto-dominates other equilibria – carries over in a natural way to the subgames of $\Gamma(\delta)$. If players are susceptible to strategic risk, they are so at all nodes of the game. Hence, a non-risky equilibrium path supported by risky out-of-equilibrium (punishment) may not be considered a sufficiently 'safe' equilibrium. Players who are concerned about strategic risk may find these concerns aggravated after having observed deviations in the past.

A subgame $\Gamma^h(s)$ of $\Gamma(\delta)$ is characterized by a history $h \in H$ specifying the path of stage game actions up to the period where the subgame starts. Whether a co-operation equilibrium φ^h is risky and its riskiness $\rho^h(s,\varphi)$ restricted to $\Gamma^h(s)$ are defined equivalently by comparing Nash-products in the corresponding formation $\Gamma^h_{\varphi}(s)$, hence we can introduce the following refinement.

Definition 5 *A co-operation equilibrium* $\varphi \in \Phi(s)$ *is called* risk perfect *iff its restriction to any subgame is not risky, wherever this is defined. Formally:*

$$\rho^h(s,\varphi) \leq 0 \quad \forall h \in H.$$

It is easy to recognize that the grim trigger equilibrium ψ is risk perfect whenever it is not risky. After any deviation ψ the stage game equilibrium is played forever, which is perfectly safe at any later instant. Hence, the condition $\delta \geq \delta^*$ also guarantees that at least one risk perfect co-operation equilibrium exists.

Which other equilibria are risk perfect? To give sufficient conditions for risk perfection we restrict attention to *simple strategies* as defined by Abreu (1988). In the 2-player repeated Prisoner's Dilemma a simple strategy for player i is specified by 3 paths, the initial path π^0 and a punishment path π^j for every player $j = 1, 2$. A punishment path specifies what is played if player j deviates from the initial path or any ongoing punishment path. If no player deviates or both players deviate simultaneously a simple strategy specifies to proceed along the ongoing path.[16] As Abreu showed, every perfect equilibrium outcome can be supported by a perfect equilibrium in simple strategies.

Definition 6 *We call a punishment path π^j of a simple strategy in the repeated Prisoner's Dilemma a* monotonous restitution *if (i) no player ever switches from c to d along the path (monotony) and (ii) the punishing party $i \neq j$ never switches from d to c before the reneging party j does (restitution).*

A monotonous restitution after a deviation of, say, player 1 always takes the form

$$\pi^1 = \left(\underbrace{(d,d), ..., (d,d)}_{\text{punishment phase: } T_1 \text{ periods}}, \underbrace{(c,d), ..., (c,d)}_{\text{restitution phase: } \tau_1 \text{ periods}}, \underbrace{(c,c), (c,c), ...}_{\text{co-operation phase}} \right).$$

In a monotonous punishment path a player who starts to co-operate will co-operate forever. For example, the path $\pi^1 = ((d,d), (c,d), (d,d), (c,c), (c,c), ...)$ is clearly not monotonous, and $\pi^1 = ((d,d), (d,c), (c,c), ...)$ is not a restitution since the deviating player 1 starts co-operating later, gaining again instead of (weakly) recompensing his opponent. Monotonous restitutions include most punishments used in applications, among which:

- Grim trigger, $T_i = \infty$:
$$\pi^1 = \pi^2 = ((d,d), (d,d), ...)$$
- Tit for tat, $T_i = 0, \tau_i = 1$:
$$\begin{aligned} \pi^1 &= ((c,d), (c,c), (c,c), ...) \\ \pi^2 &= ((d,c), (c,c), (c,c), ...) \end{aligned}$$
- T-periods "defection wars" or 0-restitution, $T > 0, \tau_i = 0$:
$$\pi^1 = \pi^2 = \left(\underbrace{(d,d), ..., (d,d)}_{T \text{ periods}}, (c,c), (c,c), ... \right)$$

---

[16] To avoid introducing further notation we do not provide a formal definition of simple strategies and optimal penal codes. The details are well known, and we do not need them here; see Abreu (1988).

- Renegotiation-proof "repentance" strategies, $T = 0, \tau_i > 0$:

$$\begin{aligned}\pi^1 &= \left(\underbrace{(c,d),...,(c,d)}_{\tau \text{ periods}},(c,c),(c,c),...\right)\\ \pi^2 &= \left(\underbrace{(d,c),...,(d,c)}_{\tau \text{ periods}},(c,c),(c,c),...\right).\end{aligned}$$

We can now state the following.

Theorem 2 *Consider the discounted infinitely repeated Prisoner's Dilemma* $\Gamma(\delta)$*. Let* φ *be a subgame perfect co-operation equilibrium in simple strategies with monotonous restitution punishment paths that is not risky. Then* φ *is risk perfect.*

Proof. To see that a subgame perfect simple strategy equilibrium φ with monotonous restitution punishment paths that is not risky is also risk perfect we take advantage of the simple strategy concept. The subgame starting from any period in any future co-operation phase is equivalent to the initial path π^0. Hence assuming that on the initial path π^0 it is $\rho(s,\varphi) \leq 0$, we only need to verify that the same holds for all subgames $\Gamma^h_\varphi(s)$ starting within the monotonous restitution punishment paths π^i. To do this, we first identify some "critical" subgames, such that if $\rho(s,\varphi) \leq 0$ for that subgame, then $\rho(s,\varphi) \leq 0$ for all other subgames beginning in π^i. Then we verify riskiness for the critical subgames.

Consider the monotonous restitution

$$\pi^1 = \left(\underbrace{(d,d),...,(d,d)}_{\text{punishment phase: } T_1 \text{ periods}}, \underbrace{(c,d),...,(c,d)}_{\text{restitution phase: } \tau_1 \text{ periods}}, \underbrace{(c,c),(c,c),...}_{\text{co-operation phase}}\right)$$

where after $T_1 \geq 0$ periods of mutual non-co-operation the formerly defecting party "reimburses" the punishing party by unilaterally co-operating for τ_1 periods, with $\tau_1 \geq 0$. We called the first phase "punishment phase" and the second one "restitution phase". We now distinguish between the two cases (i) strict restitution: $\tau_1 \geq 1$ and (ii) 0-restitution: $\tau_1 = 0$.

Case (i) "Strict restitution": For $\tau_1 \geq 1$ the critical subgame starts at the beginning of the restitution phase in period $T_1 + 1$ of the monotonous restitution π^1. To see this, note that in subgames starting within the punishment phase sticking to equilibrium strategies is strictly less risky than in subgames starting during the restitution phase, since playing d involves no risk and players discount future (risk). Now note that the risk for player 1 involved in playing c at the beginning of the restitution phase $(T + 1)$ is at least as large as in subsequent periods (T_1+2) to $(T_1+\tau_1)$. Therefore, for $\tau_1 \geq 1$ it remains to show that the riskiness property is satisfied in the critical subgame beginning period T_1+1 of the monotonous restitution π^1, denoted by $\Gamma_{\varphi}^{h_1}(s)$. By our definition of riskiness we have to look at the $2\times 2-$formation $\Gamma_{\varphi}^{*h_1}(s)$ of $\Gamma_{\varphi}^{h_1}(s)$ where each player i only compares playing the equilibrium strategy $\varphi_i^{*h_1}$ and $\omega_i^{*h_1}$ (play always d). By subgame perfection this formation again must have two equilibria φ^{*h_1} and ω^{*h_1} induced by φ and ω. Next, note that a strict restitution phase $\tau_1 \geq 1$ prescribes that in φ^{*h_1} player 1 starts to reimburse player 2. If player 1, however, plays $\omega_1^{*h_1}$ and fails to do so, both players obtain the same payoff as in equilibrium ω^{*h_1} of the formation $\Gamma_{\varphi}^{*h_1}(s)$. Hence, ω^{*h_1} is a weak equilibrium since player 2 is indifferent between $\varphi_2^{*h_1}$ and $\omega_2^{*h_1}$ if player 1 plays $\omega_2^{*h_1}$. This implies that the Nash product of ω^{*h_1} is 0 and therefore is φ^{*h_1} not risky.

Case (ii): "0-restitution phase": For the same reason as in the strict restitution phase the critical subgame is the one that starts at the beginning of the co-operation phase in the period T_1+1 of the monotonous restitution π^1. But this subgame is equivalent to the initial game starting in π^0 where the equilibrium φ is not risky by assumption. This concludes the proof. ■

Hence, for most punishment strategies used in the literature (monotone restitutions), checking that the initial equilibrium path is not risky $(\rho(s,\varphi) \leq 0)$ is sufficient to guarantee risk perfection $(\rho^h(s,\varphi) \leq 0 \ \ \forall h \in H)$. Regarding other equilibria, one has to check case by case. Consider, for example, an equilibrium in non-simple strategies where the first deviation from the equilibrium outcome path is punished differently than further deviations. Let φ be a co-operation equilibrium where punishment paths after the first deviation of player j, denoted by π_1^j, are given by

$$\pi_1^1 = \pi_1^2 = \left(\underbrace{(d,d),...,(d,d)}_{T^1 \text{ periods}}, (c,c), (c,c), ... \right),$$

with $T^1 > 1$. Now let $k(h)$ be the number of previous deviations from equilibrium behavior in history h, and suppose equilibrium strategies prescribe, for any further deviation $k>1$,

$$\pi_k^1 = \pi_k^2 = \left(\underbrace{(d,d)}_{T^k=1}, (c,c), (c,c), ... \right),$$

i.e. defecting just once before returning to co-operation. Riskiness at the start of the game can be kept small by increasing T^1, while the subgame starting after these T^1 periods of punishment is subject to higher risk. It is easy to check that $T^1 > 1$ implies $\rho(s,\varphi) < \rho^k(s,\varphi)$ for $k>1$. Hence, if parameters are such that $\rho(s,\varphi) \leq 0 < \rho^k(s,\varphi)$

the equilibrium is risk undominated but not risk perfect (it is risk perfect iff $\rho^k(s,\varphi) \leq 0$).

5 Extensions

5.1 Parameter Asymmetries

In many applications – for example in models of customer-client or employer-employee relationships – the PD-game under consideration is asymmetric in payoff parameters or even in discount factors. While the qualitative structure of all our results remains unaffected by asymmetry, the analysis of this more general case yields additional structure. Moreover, for applications the more general formula for δ^* can be of practical value.

Consider the stage game Γ given by

Γ	c	d
c	c_2 c_1	b_2 a_1
d	a_2 b_1	d_2 d_1

,

with $b_i > c_i > d_i > a_i$ for $i = 1,2$ and $c_1 + c_2 > \max[b_1 + a_2, a_1 + b_2]$. Denote accordingly by $s = (a_1, b_1, c_1, d_1, \delta_1, a_2, b_2, c_2, d_2, \delta_2)$ the set of exogenous parameters for the infinitely repeated discounted game $\Gamma(s)$ where discount factors and payoffs may depend on the player.[17] Similarly, define the related parameter space supporting strict co-operation equilibria of $\Gamma(\delta)$ as

$$S := \left\{ s \;\middle|\; \begin{array}{c} b_i > c_i > d_i > a_i \text{ for } i = 1,2 \text{ and} \\ c_1 + c_2 > \max[b_1 + a_2, a_1 + b_2] \text{ and} \\ \underline{\delta}_i \equiv \frac{b_i - c_i}{b_i - d_i} < \delta_i < 1 \text{ for } i = 1,2 \end{array} \right\}.$$

As before the condition

$$\underline{\rho}(s) = (d_1 - a_1)(d_2 - a_2) - \left(\frac{c_1}{1-\delta_1} - b_1 - \delta_1 \frac{d_1}{1-\delta_1}\right)\left(\frac{c_2}{1-\delta_2} - b_2 - \delta_2 \frac{d_2}{1-\delta_2}\right) > 0$$

---

[17] We do not study here the consequences of trading effects among players with different discount factors. It is well known (see for example Lehrer and Pauzner 1999) that there are positive gains from trade between an impatient and a patient player that enhance the set of equilibrium payoffs. Therefore, the co-operation equilibrium is not neccessarily the most efficient equilibrium and is not neccessarily the "natural candidate" to compare with all-defect for assessing risk. However, in this section we only compare co-operation equilibria for expositional convenience. In the following section we will see that comparing other efficient equilibria tends to aggravate the risk dominance problem.

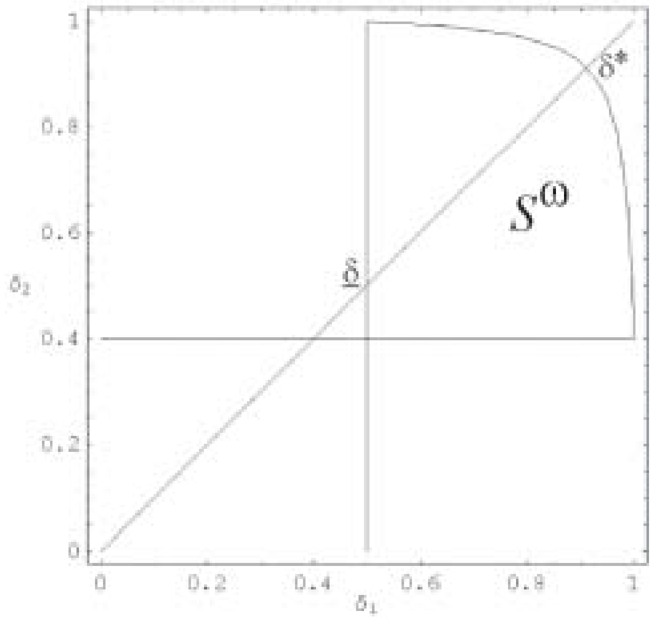

Figure 1: S^{ω} for $a_1 = -5, a_2 = -20, d_1 = 6, d_2 = 5, c_1 = 7, c_2 = 8, b_1 = 8, b_2 = 10$

for the lower bound on the riskiness of all co-operation equilibria characterizes the subset $S^{\omega} = \left\{s \in S \left| \underline{\rho}(s) > 0\right.\right\}$ of parameters within S such that all co-operation equilibria are risky.

Figure 1 is a projection of S^{ω} on discount factors δ_1, δ_2 for asymmetric payoff parameters. The diagonal represents the case of symmetric discount factors $\delta \equiv \delta_1 = \delta_2$ and contains the intersection with S^{ω} where all co-operation equilibria of $\Gamma(s)$ are risky. For the latter case of asymmetric payoff parameters and symmetric discount rates δ^* solves the quadratic equation $\underline{\rho}(s) = 0$ and $\underline{\delta}$ and δ^* are then given by

$$\begin{aligned} \underline{\delta} &= \max\left\{\frac{b_1 - c_1}{b_1 - d_1}, \frac{b_2 - c_2}{b_2 - d_2}\right\}, \\ \delta^* &= \frac{Y + Z}{2W} + \sqrt{\left(\frac{Y + Z}{2W}\right)^2 - \frac{X}{W}}, \end{aligned}$$

where

$$\begin{aligned} X &= (d_1 - a_1)(d_2 - a_2) - (b_1 - c_1)(b_2 - c_2), \\ Y &= (d_1 - a_1)(d_2 - a_2) - (b_1 - d_1)(b_2 - c_2), \\ Z &= (d_1 - a_1)(d_2 - a_2) - (b_1 - c_1)(b_2 - d_2), \text{ and} \\ W &= (d_1 - a_1)(d_2 - a_2) - (b_1 - d_1)(b_2 - d_2). \end{aligned}$$

The asymmetric case reveals in particular that strategic risk is a genuinely bilateral phenomenon, to be distinguished carefully from individual risk. If, for example, $d_i \to a_i$ for one player, say player 1, then the riskiness of co-operation equilibria goes to 0.[18] In that case player 1 is not subject to risk when choosing to co-operate. Since both players are aware of the fact that one player risks nothing by picking the Pareto-superior equilibrium, the sort of mutual doubt underlying strategic risk considerations disappears altogether. This is for example the case in the "trust game" – a sequential one-sided Prisoner's Dilemma where only one player can cheat. The truly bilateral essence of strategic risk can also be appreciated by noting that even in the asymmetric case there are unique values of ρ and δ^* identical for both players, while generically $\underline{\delta}_1 \neq \underline{\delta}_2$ and (in the absence of payoff transfers) one has to take the $\max\{\underline{\delta}_1, \underline{\delta}_2\}$ to identify $\underline{\delta}$.

5.2 Risky Efficient Equilibria

Does Prisoners' Other Dilemma aggravate or alleviate for other efficient equilibria, i.e. when payoffs are distributed asymmetrically among players on the Pareto frontier? Consider our symmetric PD supergame $\Gamma(\delta)$, and denote by $\theta(x)$ the efficient equilibrium yielding averaged asymmetric per-period payoffs $c - x, c + \frac{b-c}{c-a}x$ with $c - d \geq x \geq 0$. The following proposition is a reformulation of proposition 2 for these equilibria.

Proposition 3 *For every $\delta < 1$ there exist payoff parameters λ with $s = (\lambda, \delta) \in S$ such that all equilibria $\theta(x)$ supporting the same payoffs are risky. Moreover, for any given riskiness $\rho > 0$ there exist payoff parameters λ with $s = (\lambda, \delta) \in S$ such that all equilibria $\theta(x)$ have at least the riskiness ρ.*

Proof. The riskiness of $\theta = \theta(x)$ is given by

$$\begin{aligned}\rho(s,\theta) \quad : \quad &= v_1(\theta)\, v_2(\theta) - u_1(\theta)\, u_2(\theta) \\ &= \left(\frac{d}{1-\delta} - a - \delta V_{1\varphi}\right)\left(\frac{d}{1-\delta} - a - \delta V_{2\varphi}\right) \\ &\quad - \left(\frac{c-x}{1-\delta} - b - \delta V_{1\omega}\right)\left(\frac{c + \frac{b-c}{c-a}x}{1-\delta} - b - \delta V_{2\omega}\right)\end{aligned}$$

implying that $\rho(s,\theta) \to \infty$ for $a \to -\infty$. ■

The following proposition shows that moving away from symmetric co-operation along the Pareto frontier increases the riskiness if off-equilibrium punishments are kept constant and symmetric (as, for example, in 'grim trigger' and 'tit for tat').

---

[18] Note that if $d_i = a_i$ for a player, the set Φ of strict cooperation equilibria is empty by definition since $v_i = 0$.

Proposition 4 *Let φ be a co-operation equilibrium, and $\theta(x)$ be an equilibrium on the Pareto frontier yielding averaged asymmetric per-period payoffs $c - x, c + \frac{b-c}{c-a}x$, with $c - d \geq x > 0$, supported by the same off-equilibrium punishments as φ. Assume further that punishments are symmetric for defecting players $V_{2\omega} = V_{1\omega}$. Then $\theta(x)$ is more risky than φ and riskiness increases with x:*

$$\rho(s, \theta(x)) - \rho(s, \varphi) > 0 \text{ and } \frac{\partial}{\partial x}(\rho(s, \theta(x)) - \rho(s, \varphi)) > 0$$

Proof.

$$\begin{aligned}
\rho(s,\theta) - \rho(s,\varphi) &= v_1(\theta) v_2(\theta) - u_1(\theta) u_2(\theta) - (v_1(\varphi) v_2(\varphi) - u_1(\varphi) u_2(\varphi)) \\
&= u_1(\varphi) u_2(\varphi) - u_1(\theta) u_2(\theta) \\
&= \left(\frac{c}{1-\delta} - b - \delta V_{1\omega}\right)\left(\frac{c}{1-\delta} - b - \delta V_{2\omega}\right) \\
&\quad - \left(\frac{c-x}{1-\delta} - b - \delta V_{1\omega}\right)\left(\frac{c + \frac{b-c}{c-a}x}{1-\delta} - b - \delta V_{2\omega}\right) \\
&= \left[\frac{c}{1-\delta} - b - \delta V_{2\omega} - \frac{b-c}{c-a}\left(\frac{c}{1-\delta} - b - \delta V_{1\omega}\right)\right]\frac{x}{1-\delta} \\
&= \left(\frac{c}{1-\delta} - b - \delta V_{1\omega}\right)\left(1 - \frac{b-c}{c-a}\right)\frac{x}{1-\delta} \\
&> 0
\end{aligned}$$

since $2c > b + a \Rightarrow 1 - \frac{b-c}{c-a} > 0$. ■

5.3 Application: Repeated Cournot Duopoly

In this section we apply previous results to a textbook example, the repeated Cournot duopoly with a linear demand function $P(Q) = \alpha - \beta Q$, where $Q = q_1 + q_2$ is total quantity and symmetric constant marginal costs are denoted by C. While this is a continuous strategy stage game, some relevant strategic aspects are captured by a substructure similar to a Prisoner's Dilemma.[19] Hence we consider a reduced stage game assuming that duopolists restrict their attention to

- choosing Cournot-quantities $q^d = \frac{\alpha - C}{3\beta}$ (defect) or
- choosing symmetric joint monopoly quantities $q^c = \frac{\alpha - C}{4\beta}$ (collude).

---

[19] A relevant difference is the possibility to punish more severely in the continuous strategy repeated game. This enlarges the feasable range of differentiating continuation payoffs which – as we pointed out in the previous section – can mitigate the problem to some extent.

According to our earlier notation, we denote the unique Nash equilibrium of this PD stage game – the Cournot-equilibrium – by $\omega = \left(q^d, q^d\right)$. Calculating the reaction functions and related profits yields payoff parameters $\lambda = (a, b, c, d) = (54X, 81X, 72X, 64X)$ with $X = \frac{(\alpha - C)^2}{576\beta}$. Normalizing demand parameters so that $X = 1$ we obtain the following bi-matrix form for the reduced Cournot PD-stage game

	q^c	q^d
q^c	72 72	81 54
q^d	54 81	64 64

.

The theorem shows that for $\delta \in (\underline{\delta}, \delta^*) = \left(\frac{9}{17}, \frac{19}{27}\right) \approx (.53, .7)$ there exists no collusive equilibrium of the discounted repeated Cournot duopoly PD-game that is not risky; and that for $\delta^* \leq \delta < 1$ there always exist some collusive equilibria that are risky by the Cournot-Nash equilibrium.

6 Appendix

The bicentric prior and the tracing procedure in the infinitely repeated Prisoner's Dilemma The *bicentric prior* – loosely speaking – is a mixed strategy profile reflecting players' initial beliefs on which of two equilibria under comparison should be played. More precisely, say, player $-i$ attaches subjective probability $(z, 1 - z)$ to strategies $\varphi_{-i}, \omega_{-i}$. Since player i has no idea of player $-i$'s subjective probability HS invoke the principle of insufficient reason and let player i's bicentric prior $p_i, 1 - p_i$ be defined as a best response on the mixture $(Z, 1 - Z)$ where Z is a random variable uniformly distributed over $[0, 1]$. Since generally the bicentric prior is not an equilibrium in itself HS define the *tracing procedure* such as to readjust beliefs to obtain the risk-dominant equilibrium. To do this precisely, define the payoffs of a parametrized family of auxiliary games denoted by $\Gamma^t(\delta, p)$ as follows. Consider strategy profile $\xi = (\xi_1, \xi_2)$ of $\Gamma(\delta)$. Let player i assume that player $-i$ plays his bicentric prior abbreviated by p_{-i} with probability $(1 - t)$ and ξ_{-i} with probability t such that player i obtains

$$U_i^t(\xi) = (1 - t)\, U_i(\xi_i, p_{-i}) + t U_i\left(\xi_i, \xi_{-i}\right).$$

The so defined game $\Gamma^t(\delta, p)$ can be interpreted as a convex combination of the original game $\Gamma(\delta)$ and the trivial game where each player plays against his bicentric prior. Denote by $G = \{(t, \xi) \,|\xi \text{ is an equilibrium for } \Gamma^t(\delta, p)\}$ the graph of the equilibrium correspondence for the $t-$parametrized family of games $\Gamma^t(\delta, p)$.

For generic finite games it can be shown that G contains a unique distinguished curve connecting the best reply (which is the equilibrium) of $\Gamma^0(\delta, p)$ equilibrium with one of the two given equilibria (either φ or ω) in $\Gamma^1(\delta, p)$ for almost any prior p. Unfortunately, this is not anymore the case for infinite games. Therefore, for the purpose of the present paper we cannot follow the tracing procedure in order to define risk dominance.

Our criteria for an appropriate model of strategic risk in our context are (i) it must be defined for the discounted infinitely repeated Prisoner's Dilemma, (ii) it should capture the intuitive observation that payoff a enters the propensity to co-operate, (iii) be as simple as possible (Ockham's razor). HS's theory at least fails on criteria (i) and (iii) and clearly any extension of it would fail to a larger extent on (iii).

As already mentioned, it is known that for finite games with more than two strategies risk dominance within Γ_φ and risk dominance based on the bicentric prior and the tracing procedure can differ (see van Damme and Hurkens 1998, 1999; and Carlsson and van Damme 1993b). For example, it may be the case that the best reply against the bicentric prior puts some positive weight on a third strategy which is not among the two equilibria under comparison and thereby tilts the bicentric prior tracing procedure comparison to another direction than the direct comparison does. As we have seen this concern is not well defined within the repeated PD. To the contrary, we note that here we do not compare two arbitrary equilibria of the repeated PD game. Rather, we compare a co-operation equilibrium φ with the uniquely "safe" all-defection equilibrium ω. For this comparison there is a best reply to the bicentric prior that is independent of the auxiliary game parameter t. One such best reply against different mixtures parametrized by t is always either all defect ω or grim trigger denoted by ψ. There is no potential gain in playing any other co-operation strategy since grim trigger is payoff-equivalent with all other co-operation strategies against another co-operation strategy[20] but is better than any other co-operation strategy against all defect while all defect is a best response against all defect.

To see that the direct risk comparison in this paper is in line with the belief adjustment according to this constant best response to the bicentric prior (instead of the tracing procedure which is not well defined) we compute player i's bicentric prior by defining expected payoffs of responding to the joint mixture $z\varphi_{-i} + (1-z)\,\omega_{-i}$ by either of the pure strategies ψ_i or ω_i:

$$\begin{aligned}
\tilde{c}_i(z) \quad &: = U\left(\psi_i, z\varphi_{-i} + (1-z)\,\omega_{-i}\right) = \frac{zc}{1-\delta} + (1-z)\left(a + \delta\frac{d}{1-\delta}\right). \\
\tilde{d}_i(z) \quad &: = U\left(\omega_i, z\varphi_{-i} + (1-z)\,\omega_{-i}\right) = z\left(b + \delta V_{i\omega}\right) + (1-z)\,\frac{d}{1-\delta}
\end{aligned}$$

---

[20] In a co-operation equilibrium by definition no player can gain by deviating from indefinite co-operation.

Now compare these expected payoffs making use of $u_i(\varphi) = \frac{c}{1-\delta} - b - \delta V_{i\omega}$ and $v_i(\psi) = \frac{d}{1-\delta} - a - \delta \frac{d}{1-\delta} = d - a$:

$$\begin{aligned} \tilde{c}_i(z) &\geq \tilde{d}_i(z) \Leftrightarrow \\ z\left(u_i(\varphi) + v_i(\psi)\right) &\geq v_i(\psi) \Leftrightarrow \\ z &\geq \frac{v_i(\psi)}{u_i(\varphi) + v_i(\psi)}. \end{aligned}$$

Player i's bicentric prior probabilities are given by the lengths of the subintervals $[z,1]$ and $[0,z]$:

$$\begin{aligned} p_i(\psi_i) &= \frac{u_i(\varphi)}{u_i(\varphi) + v_i(\psi)} \text{ and} \\ p_i(\omega_i) &= \frac{v_i(\psi)}{u_i(\varphi) + v_i(\psi)}. \end{aligned}$$

Since the tracing procedure is not well defined we now turn to a constant (in t) best response to the bicentric prior. At the starting point $t = 0$ player i picks a best response to his bicentric prior $p_{-i} := p_{-i}\psi_{-i} + (1 - p_{-i})\,\omega_{-i}$. For any $t \in (0,1)$ responding to p_{-i} by playing φ_i or ω_i yields the payoff comparison

$$\begin{aligned} U_i(\omega_i, p_{-i}) &> U_i(\varphi_i, p_{-i}) \Leftrightarrow \\ \frac{v_i(\varphi)\, v_{-i}(\varphi)}{u_{-i}(\varphi) + v_{-i}(\psi)} &> \frac{u_i(\varphi)\, u_{-i}(\varphi)}{u_{-i}(\varphi) + v_{-i}(\psi)} \Leftrightarrow \\ v_i(\varphi)\, v_{-i}(\varphi) &> u_i(\varphi)\, u_{-i}(\varphi) \end{aligned}$$

This shows in the language of HS that $U_i(\omega_i, p_{-i}) > U_i(\varphi_i, p_{-i})$ for $i = 1, 2$ iff ω's Nash product $v_i(\varphi)\, v_{-i}(\varphi)$ is strictly larger than φ's Nash product $u_i(\varphi)\, u_{-i}(\varphi)$. In this case an equilibrium in $\Gamma^t(s,p)$ contains no positive weight on φ for any $t \in (0,1)$ since ω outperforms φ against the bicentric prior p and the constant best response p_{-i}. The converse holds for $U_i(\varphi_i, p_{-i}) > U_i(\omega_i, p_{-i})$. This shows that this form of belief adjustment corresponds exactly to our definition of risk dominance within the $2 \times 2-$game Γ_φ.

References

[1] Abreu, D. (1988) "On the Theory of Infinitely Repeated Games with Discounting", *Econometrica* 56, 383-396

[2] Aumann, R. (1990) "Nash Equilibria are not Enforceable", in *Economic Decision Making* (J.J. Gabszewitz et. al. Eds.), 201-206.

[3] Carlsson, H. and E. van Damme (1993a) "Global Games and Equilibrium Selection", *Econometrica* 61, 989-1018.

[4] Carlsson, H. and E. van Damme (1993b) "Equilibrium Selection in Stag Hunt Games", in *Frontiers in Game Theory* (ed. K.Binmore and A.Kirman)*,* Boston, M.I.T Press, 237-254

[5] van Damme, E. (1989) "Renegotiation-proof Equilibria in Repeated Prisoner's Dilemma", *Journal of Economic Theory* 47, 206-217.

[6] van Damme, E. and S. Hurkens (1998), "Endogenous Price Leadership", mimeo

[7] van Damme, E. and S. Hurkens (1999), "Endogenous Stackelberg Leadership", *Games and Economic Behavior 28, 105-29*

[8] Farrell, J. and Maskin, E. (1989), "Renegotiation in Repeated Games", *Games and Economic Behavior* 1, 327-360.

[9] Friedman, J. (1971) "A Non-cooperative Equilibrium for Supergames", *Review of Economic Studies* 38, 1-12.

[10] Fudenberg D. and Maskin, E. (1986), "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information", *Econometrica*, 54, 533-556.

[11] Fudenberg D. and Tirole, J. (1991), *Game Theory*, Cambridge, MA: M.I.T. Press.

[12] Harsanyi, J. (1995) "A New Theory of Equilibrium Selection for Games with Complete Information", *Games and Economic Behavior* 8, 91-122

[13] Harsanyi, J., and R. Selten (1988) *A General Theory of Equilibrium Selection in Games*, Boston MA: M.I.T. Press.

[14] van Huyck, J., R. Battalio and R. Beil (1990) "Tacit Coordination Games, Strategic Uncertainty, and Coordination Failure", *American Economic Review 80,* 234-248

[15] Kandori, M.; Mailath, G. and R. Rob (1993) "Learning, Mutation, and Long Run Equilibria in Games", *Econometrica* 61, 29-56

[16] Lehrer, E. and A. Pauzner (1999), "Repeated Games with Differential Time Preferences", *Econometrica 67, 393-412*

[17] Schmidt, D., R. Shupp, J.M. Walker and E. Ostrom (2003), "Playing Safe in Coordination Games: The Roles of Risk Dominance, Payoff Dominance, and History of Play", *Games and Economic Behavior* 42, 281-299

[18] Young, P. (1993), “The Evolution of Conventions”, *Econometrica* 61, 57-84.